CURVILINEAR MOTION OF AN ELLIPSOIDAL BUBBLE

A. G. Petrov

The Lagrange equations are used to investigate the curvilinear motion of an ellipsoidal bubble. At a small inclination of the minor axis of the ellipsoid to the vertical, the ellipsoid begins to oscillate about the equilibrium position and its trajectory begins to swing. Analytic expressions are presented for the oscillation frequency and for the ratio of the swing amplitude to the amplitude of the oscillations of the ellipsoid. It is assumed that the bubble has the form of an axially symmetrical ellipsoid [1].

Let a Cartesian coordinate system whose axes x_1, x_2, and x_3 are directed parallel to the principal axes of the ellipsoid be connected with an immobile coordinate system y_1, y_2, and y_3 by the relation

$$\begin{gathered} x_i = A_{ij} y_j; \quad A_{11} = \cos\varphi, \quad A_{12} = \sin\varphi, \quad A_{13} = 0 \\ A_{21} = -\sin\varphi \cos\theta, \quad A_{22} = \cos\varphi \cos\theta, \quad A_{23} = \sin\theta \\ A_{31} = \sin\varphi \cos\theta, \quad A_{32} = -\cos\varphi \sin\theta, \quad A_{33} = \cos\theta \end{gathered} \tag{1}$$

Summation over repeated indices is assumed throughout; A_{ij} is an orthogonal matrix, θ is the Euler angle between the axes x_3 and y_3, and φ is the angle between the line of nodes and the y_1 axis. By virtue of the symmetry of the ellipsoid with respect to the x_3 axis, it can be assumed that the x_1 axis lies in the horizontal plane $y_1 y_2$ and coincides with the line of nodes.

Let $y_i^{\cdot}$ and $x_i^{\cdot}$ be the components of the velocity of the center of the ellipsoid in the immobile and mobile coordinate systems, respectively. Knowing the Euler angle, we can express $x_i^{\cdot}$ in terms of $y_i^{\cdot}$ and conversely,

$$x_i^{\cdot} = A_{ij} y_j^{\cdot}, \qquad y_i^{\cdot} = A_{ji} x_j^{\cdot} \tag{2}$$

Since the moment of inertia of the ellipsoid about the x_3 axis is equal to zero, it can be assumed that the angular-velocity vector is perpendicular to the x_3 axis. The square of the angular velocity is equal to $\theta^{\cdot 2} + \sin^2\theta\, \varphi^{\cdot 2}$. The kinetic energy T of the liquid in which the rotating and deforming ellipsoid moves is equal to [1, 2]

$$\begin{gathered} T = T_0 + T_1, \quad 2T_1 = \lambda_i x_i^{\cdot 2} + I\varphi^{\cdot 2} \sin^2\theta, \quad 2T_0 = I\theta^{\cdot 2} + I_\alpha \alpha^{\cdot 2} \\ \lambda_1 = \lambda_2 = \frac{4\pi\rho}{3} l^3 \frac{1 - yB}{1 + yB}, \qquad \lambda_3 = \frac{4\pi\rho}{3} l^3 \frac{yB}{1 - yB} \\ I = \frac{4\pi\rho l^5 (3yB - 1)]}{15 (\alpha y)^{2/3} [2 - (3yB - 1)(1 + 2\alpha^2)]} \\ B = 1 - \alpha \operatorname{arcctg} \alpha, \quad y = 1 + \alpha^2, \quad \alpha / \sqrt{1 + \alpha^2} = l_3 / l_1 \end{gathered} \tag{3}$$

Here l_3 and l_1 are the lengths of the minor and major axes of the ellipsoid, l is the radius of the sphere with equivalent volume ($l^3 = l_1^2 l_3$), and $I_\alpha \dot{\alpha}^2$ is that part of the kinetic energy which is due to the deformation of the bubble. An analytic expression for I_α was derived earlier [1].

If we choose as the system of generalized coordinates $q_i = y_i$, $q_4 = \theta$, $q_5 = \varphi$, $q_6 = \alpha$, then the motion of the ellipsoidal bubble will be described by the following system of equations [1]:

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$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = \frac{4\pi}{3}\rho l^3 g\delta_{3i} - \frac{1}{2}\frac{\partial E}{\partial q_i^{\cdot}}$$

$$L = T_0 + T_1 - \sigma S, \quad \delta_{3i} = \begin{cases} 1, & i = 3 \\ 0, & i \neq 3 \end{cases} \tag{4}$$

$$S = 2\pi l^2 \frac{y^{1/3}}{\alpha^{2/3}}\left(1 + \frac{\alpha}{\sqrt{y}}\ln\frac{1+\sqrt{y}}{\alpha}\right) \quad (i = 1, 2, \ldots 6)$$

$$2T_1 = \lambda_1 y_i^{\cdot} y_i^{\cdot} + (\lambda_3 - \lambda_1) A_{3i} A_{3j} y_i^{\cdot} y_j^{\cdot} + I\varphi^{\cdot 2}\sin^2\theta$$

$$E = E_1 y_i^{\cdot} y_i^{\cdot} + (E_3 - E_1) A_{3i} A_{3j} y_i^{\cdot} y_j^{\cdot} + E_4(\theta^{\cdot 2} + \varphi^{\cdot 2}\sin^2\theta) + E_5\alpha^{\cdot 2}$$

$$(i, j = 1, 2, 3)$$

Here E is the energy dissipation, which in the case of large Reynolds numbers R is calculated in terms of the velocity-field potential [1].

The stationary solution of equations (4), corresponding to a vertical rise of the bubble with constant velocity (y_1, y_2, θ, φ are equal to zero, $\alpha = \alpha_0$) was obtained earlier [1].

At a small deviation of the generalized coordinates from their equilibrium values, damped oscillations will take place.

Estimates show that, in order of magnitude, the damping time exceeds the period of the oscillations by R times. Therefore, the processes occurring in times much shorter than the damping time can be described by the Lagrange equation without allowance for the viscous terms

$$\frac{d}{dt}\frac{\partial L}{\partial q_i^{\cdot}} - \frac{\partial L}{\partial q_i} = 0 \tag{5}$$

The first four coordinates are cyclic, as a result of which we have the angular-momentum conservation law and three momentum conservation laws

$$I\varphi^{\cdot}\sin^2\theta = M, \quad \lambda_i y_i^{\cdot} + (\lambda_3 - \lambda_1) A_{3i} A_{3j} y_j^{\cdot} = P_i \tag{6}$$

The immobile system of coordinates can always be chosen such that the momentum vector P is directed along the y_3 axis, so that we can put $P_i = P\delta_{3i}$. Solving the system (6) with respect to $y_i^{\cdot}$ and $\varphi^{\cdot}$, we obtain

$$y_1^{\cdot} = P(\lambda_3^{-1} - \lambda_1^{-1})\sin\varphi\sin\theta\cos\theta, \quad y_2^{\cdot} = P(\lambda_3^{-1} - \lambda_1^{-1})\cos\varphi\sin\theta\cos\theta$$

$$y_3^{\cdot} = P(\lambda_3^{-1}\cos^2\theta + \lambda_1^{-1}\sin^2\theta), \quad \varphi^{\cdot} = M/(I\sin^2\theta) \tag{7}$$

Given P and M, the system (5) is equivalent to a dynamic system with kinetic energy T_0 and potential energy V equal to

$$V = \sigma S + {}^1/_2(P_i y_i^{\cdot} + M\varphi^{\cdot}) = \sigma S + {}^1/_2 P^2(\lambda_3^{-1}\cos^2\theta + \lambda_1^{-1}\sin^2\theta) + M^2(2I\sin^2\theta)^{-1} \tag{8}$$

Thus, the equations of motion are

$$\frac{d}{dt}\frac{\partial T_0}{\partial \theta^{\cdot}} = -\frac{\partial V}{\partial \theta}, \quad \frac{d}{dt}\frac{\partial T_0}{\partial \alpha^{\cdot}} = \frac{\partial}{\partial \alpha}(T_0 - V) \tag{9}$$

At small deviations of θ and α from their equilibrium values, Eqs. (7) and (9) become linearized

$$y_1^{\cdot} = P\theta(\lambda_3^{-1} - \lambda_1^{-1})\sin\varphi, \quad y_2^{\cdot} = P\theta(\lambda_3^{-1} - \lambda_1^{-1})\cos\varphi$$

$$y_3^{\cdot} = P\lambda_3^{-1}, \quad \varphi^{\cdot} I\theta^2 = M, \quad I\theta^{\cdot\cdot} = \frac{M^2}{I\theta^3} - P^2(\lambda_1^{-1} - \lambda_3^{-1})\theta, \tag{10}$$

$$I_\alpha\alpha^{\cdot\cdot} = -\frac{\partial}{\partial\alpha}\left(\sigma S + \frac{P}{2\lambda_3} + \frac{M^2}{2I\theta^2}\right)$$

The rate of rise of the bubble $u = y_3^{\cdot}$ is determined from the balance between the Archimedes force and the viscous-resistance force. The last equation of (10) determines the deviation of the degree of deformation from the equilibrium value. The corresponding formulas for the deformation frequency Ω_α were obtained earlier [1]. At M = 0, the next to the last equation in (10) shows that θ oscillates at a frequency Ω_θ. Since $\varphi^{\cdot} = 0$ in this case, the bubble-motion trajectory is a plane curve, and it can be assumed that $\varphi = 0$. Thus, equations (10) have the following solution:

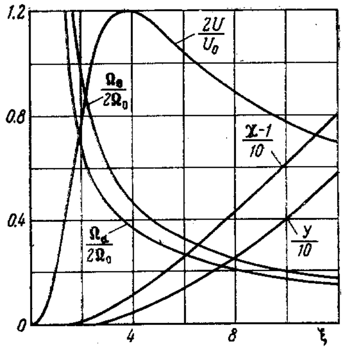

Fig. 1

$$\alpha - \alpha_0 = \Delta\alpha \cos(\Omega_\alpha t + \gamma), \quad \theta = \theta_0 \sin(\Omega_\theta t)$$

$$y_2 = l\theta_0 Y \cos(\Omega_\theta t), \quad y_1 = 0$$

$$\Omega_\theta = u\left[\frac{(\lambda_3 - \lambda_1)\lambda_3}{\lambda_1 I}\right]^{1/2}, \qquad Y^2 = \frac{(\lambda_3 - \lambda_1) I}{\lambda_1 \lambda_3 l^2} \tag{11}$$

It follows from (11) that the horizontal displacement of the center of the ellipsoid y_2 leads in phase, by $\pi/2$, the inclination angle of the minor axis of the ellipsoid. If $M \neq 0$, then there exists a solution of (10) at which $\theta^{\cdot} = 0$ and $\alpha^{\cdot} = 0$, the ellipsoid moving along a helical line with amplitude $l\theta_0 Y$ and rotation frequency Ω_θ.

The motion of a solid in an unbounded ideal incompressible liquid was investigated by many scientists, including Chaplygin, Lyapunov, Kirchhoff, Clebsch, and others. Formulas (5)-(10) agree with the corresponding results given in the monographs [2-4].

The diagram shows plots of the functions $\chi = l_3/l_1$, u/u_0, Ω_α/Ω_0 against the dimensionless radius of the equivalent-volume sphere $\xi = l/l_0$. At a small degree of deformation, $\chi - 1 \ll 1$, the corresponding functions take the form

$$\chi - 1 = \frac{\xi^5}{288}, \quad \frac{u}{u_0} = \frac{\xi^2}{9}, \quad \frac{\Omega_\alpha}{\Omega_0} = \frac{2\sqrt{3}}{\xi^{3/2}}, \quad \frac{\Omega_\theta}{\Omega_0} = \frac{2\sqrt{7}}{\xi^{3/2}}$$

$$Y = \frac{\xi^{15/2}}{2880\sqrt{7}}, \quad l_0^5 = \frac{\sigma\nu^2}{\rho g^2}, \quad u_0^5 = \frac{\sigma^2 g}{\rho^2 \nu}, \quad \Omega_0 = \frac{u_0}{l_0}$$

It is seen from the figure that the rate of rise of the bubble has at the point $\xi = 3.84$ a maximum equal to 0.6005, and the frequencies Ω_α and Ω_θ coincide in order of magnitude with the frequency of the capillary oscillations of a sphere of radius l. The function $Y(\xi)$ increases rapidly with increasing dimension of the bubble and, therefore, for a large bubble relatively small deviations of the ellipsoid axis from the vertical cause large horizontal displacements of the center of the ellipsoid.

It should be noted that the results are valid at a large Reynolds number and, on the other hand, the degree of deformation of the ellipsoid χ should not exceed approximately five, for otherwise the approximation of the potential flow-around is not valid [5]. The dimensionless radius of the bubble must therefore satisfy the following conditions:

$$\left(\frac{\sigma^3}{\rho^3 g \nu^4}\right)^{1/5} \xi \frac{u}{u_0} \gg 1, \qquad \xi \lesssim 8$$

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